

Simple Systems with Anomalous Dissipation and Energy Cascade

June 1st, 2006

**Jonathan C. Mattingly¹, Toufic Suidan², and
Eric Vanden-Eijnden³**

¹ Department of Mathematics and CNCS, Duke University, Durham, NC 27708, USA. Email: jonm@math.duke.edu

² Mathematics Department, University of California, Santa Cruz, CA 95064, USA. Email: tsuidan@ucsc.edu

³ Courant Institute, New York University, New York, NY 10012, USA. Email: eve2@cims.nyu.edu

Abstract

We analyze a class of dynamical systems of the type

$$\dot{a}_n(t) = c_{n-1}a_{n-1}(t) - c_n a_{n+1}(t) + f_n(t), \quad n \in \mathbb{N}, \quad a_0 = 0,$$

where $f_n(t)$ is a forcing term with $f_n(t) \neq 0$ only for $n \leq n_* < \infty$ and the coupling coefficients c_n satisfy a condition ensuring the formal conservation of energy $\frac{1}{2} \sum_n |a_n(t)|^2$. Despite being formally conservative, we show that these dynamical systems support dissipative solutions (suitably defined) and, as a result, may admit unique (statistical) steady states when the forcing term $f_n(t)$ is nonzero. This claim is demonstrated via the complete characterization of the solutions of the system above for specific choices of the coupling coefficients c_n . The mechanism of anomalous dissipations is shown to arise via a cascade of the energy towards the modes with higher n ; this is responsible for solutions with interesting energy spectra, namely $\mathbb{E}|a_n|^2$ scales as $n^{-\alpha}$ as $n \rightarrow \infty$. Here the exponents α depend on the coupling coefficients c_n and \mathbb{E} denotes expectation with respect to the equilibrium measure. This is reminiscent of the conjectured properties of the solutions of the Navier-Stokes equations in the inviscid limit and their accepted relationship with fully developed turbulence. Hence, these simple models illustrate some of the heuristic ideas that have been advanced to characterize turbulence, similar in that respect to the random passive scalar or random Burgers equation, but even simpler and fully solvable.

1 Introduction and main results: Life starts after blow-up

So little is understood about hydrodynamic turbulence that there is not even consensus on what it is. However, most physicists would agree on the following heuristic picture which has emerged from the works Kolmogorov, Onsager, Richardson, etc [Fri95]. In this picture, (fully developed) turbulence refers to the idealized state of an incompressible fluid described by the Navier-Stokes equations in the limit of vanishing molecular viscosity. In this limit, the Navier-Stokes equations formally reduces to the Euler equations, and the turbulent solutions should be the most regular solutions of the Euler

equations which dissipate energy. This is referred to as anomalous dissipation and is best visualized in the Fourier representation. There it corresponds to a cascade of energy from the small wavenumbers (large spatial scales) where energy is injected (either via the initial condition or by a forcing term in the equation) towards larger and larger wavenumbers (smaller and smaller scales), up to infinity where energy should eventually be dissipated. It is also believed that the cascade of energy implies that the energy spectrum of the turbulent solutions have a power law decay in the wavenumber whose rate can be deduced by dimensional analysis and is $\frac{5}{3}$ in three-dimension of space.

Turbulence theory (as we shall refer to the heuristic picture above) also discusses more advanced and more controversial topics such as intermittency. But, without even going into those, most mathematicians would agree that a rigorous confirmation of the basic predictions of turbulence theory is already a tremendous challenge. The best known results on the Navier-Stokes and Euler equations which corroborate the above were obtained in [CET94, Eyi01, DR00]. These works only indicate that turbulence theory is not blatantly inconsistent. Simpler models, such as randomly forced Burgers equation or Kraichnan's model of passive scalar advection (see e.g. [E01, FGV01] for reviews), have also been used to demonstrate that parts of turbulence theory make sense in terms of anomalous dissipation of the weak solutions of the inviscid Burgers equation and the spectrum of energy of the solutions that this implies. Even these simple models remain surprisingly complicated to analyze and a full characterization of the statistical properties of their solutions is still lacking.

One of the purposes of the present paper is to illustrate turbulence theory on even simpler models. Many (if not most) of the realistic features have been neglected in our models. Yet, the models possess a rich range of behaviors which depend on the details of the interactions. They provide a simple class of exactly solvable models which can be useful in understanding the inner workings of some energy transfer mechanisms. The solutions of these models are also consistent with much of the claims of turbulence theory. In a way, they offer a setting for the skeptical mathematician to understand the motivation behind these claims, and if this paper succeed in doing this, we will have achieved our main goal.

Next, we introduce the models that we will investigate and we summarize the principal results of the paper. As we will see, the most interesting and meaningful solutions of these models are solutions which have blown-up, such that they have become infinite in some norm. This justifies our claim that "life starts after blow-up": Disregarding these solutions as nonsensical, as one may be tempted to do at first sight, would, in fact, completely miss the most interesting phenomena displayed by the models.

1.1 A linear shell model

Consider the equation

$$\dot{a}_n(t) = c[(n-1)a_{n-1}(t) - na_{n+1}(t)]$$

for $n \in \mathbb{N}$ with the boundary condition $a_0(t) = 0$ for all t . If $c > 0$, we can rescale time to fix $c = 1$; observe also that if a_n satisfies the equations with the parameter $c < 0$ then $\hat{a}_n(t) = (-1)^{n+1}a_n(t)$ satisfies the equations with parameter $|c|$.

In light of these considerations, we set $c = 1$ and focus our attention on

$$\dot{a}_n(t) = (n-1)a_{n-1}(t) - na_{n+1}(t) \tag{1.1}$$

for $n \in \mathbb{N}$ with the boundary condition $a_0(t) = 0$ for all t .

Although we will see that this calculation is not always correct, on the formal level one has that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \sum_{n=1}^{\infty} |a_n(t)|^2 &= \sum_{n=1}^{\infty} [(n-1)a_{n-1}(t)a_n(t) - na_{n+1}(t)a_n(t)] \\ &= \sum_{n=1}^{\infty} na_n(t)a_{n+1}(t) - \sum_{n=1}^{\infty} na_n(t)a_{n+1}(t) = 0. \end{aligned} \quad (1.2)$$

The second equality is only formal as it assumes that the sum $\sum_{n=1}^{\infty} na_n(t)a_{n+1}(t)$ is finite and absolutely convergent. To understand this further, consider the evolution of the partial sum $\sum_{n \leq N} |a_n(t)|^2$. For $N \in \mathbb{N}$,

$$\frac{1}{2} \frac{d}{dt} \sum_{n \leq N} |a_n(t)|^2 = -Na_N(t)a_{N+1}(t). \quad (1.3)$$

The validity of (1.2) necessitates

$$\lim_{N \rightarrow \infty} Na_N(t)a_{N+1}(t) = 0. \quad (1.4)$$

If this condition is not satisfied, then the formal manipulation in (1.2) does *not* hold and the seemingly conservative coupling term in (1.1) may become a source of anomalous dissipation. We make the concept of anomalous dissipation precise in Section 2. But, roughly speaking, it is when seemingly conservative terms have a dissipative effect on the system.

In the context of equation (1.1), anomalous dissipation seems to require that the limit as $N \rightarrow \infty$ of the right hand side of (1.3) be negative. In other words, equation (1.1) is dissipative at time t if

$$\liminf_{n \rightarrow \infty} na_n(t)a_{n+1}(t) > 0. \quad (1.5)$$

If we make the reasonable assumption that $\lim_n a_{n+1}/a_n \in (0, \infty)$, then from (1.5) the solution of equation (1.1) will be dissipative with a finite dissipation rate provided that

$$a_n(t) \sim \frac{1}{\sqrt{n}} \quad \text{as } n \rightarrow \infty.^1 \quad (1.6)$$

At this point some readers may be skeptical since one typically considers equations like (1.1) with initial data in ℓ^2 , the space of square-summable sequences. However, we will see (Theorem 3.1 in Section 3) that equation (1.1) has solutions which exist for all time provided

$$\limsup_{n \rightarrow \infty} |a_n(0)|^{1/n} \leq 1.$$

This condition admits a large class of initial conditions including those which scale as (1.6). In Section 3, we will also see that (1.1) possesses a wide verity of behavior including conservative, dissipative, and explosive solutions.

It might be tempting to dismiss these non-conservative solutions as non-physical solution arising from pathological data. We now discuss why is not the case.

¹We say that $f(m) \sim g(m)$ as $m \rightarrow \infty$ if their exists an m_1 and $c \geq 1$ so that if $m > m_1$ then $\frac{1}{c}g(m) \leq f(m) \leq cg(m)$. Similarly, we say that $f(m) \sim g(m)$ as $m \rightarrow m_0$ if $\lim f(m)/g(m) = 1$ as $m \rightarrow m_0$.

Consider equation (1.1) with a white-noise forcing in the first coordinate:

$$\dot{a}_n(t) = (n-1)a_{n-1}(t) - na_{n+1}(t) + \mathbf{1}_{n=1}\dot{W}(t). \quad (1.7)$$

where $\mathbf{1}_{n=m}$ is 1 if $n = m$ and 0 otherwise, and $W(t)$ denotes a standard Brownian motion, i.e. Gaussian process with mean zero and covariance $\mathbb{E}W(t)W(s) = \min(t, s)$. If one were to accept the formal calculations in (1.2), showing energy conservation, then

$$\mathbb{E} \sum_{n=1}^{\infty} |a_n(t)|^2 = \mathbb{E} \sum_{n=1}^{\infty} |a_n(0)|^2 + t$$

if the energy is initially finite. Hence, in the forced system energy seems to grow linearly with time and at $t = \infty$ one expects the system to have infinite energy. These solutions which “blow-up” (in the sense that they have infinite energy) are the most interesting and relevant. In light of the discussion above, one might expect that the energy of the system would grow to be infinite and arrange the a_n so that the calculation in (1.2) is not valid since the sum is not rearrangeable. Onsager would then predict that the system would evolve to the state in which the a_n decayed as fast as possible but still dissipated energy in the sense that (1.5) holds. The reasoning which leads to (1.6) strongly suggests that the $|a_n|$ should scale as $1/\sqrt{n}$. In fact, if the system is to reach some equilibrium the effect of the dissipation must exactly balance that of the forcing. Specifically, in the stochastic setting when the forcing is $\dot{W}(t)\mathbf{1}_{n=1}$, $\lim_n n \mathbb{E}(a_n(t)a_{n+1}(t)) \rightarrow 1$ as $t \rightarrow \infty$.

All of these conclusion turn out to be correct. In particular, in Section 6 we prove that if $\sum_n |a_n(0)|^2 < \infty$ then the solutions converge to a unique random variable $a^{**} = (a_1^{**}, a_2^{**}, \dots)$ which is Gaussian with mean zero and whose distribution is the unique stationary measure for the system. Furthermore, this equilibrium state has a structure which is consistent with the anticipated $1/\sqrt{n}$ scaling:

$$\lim_{t \rightarrow \infty} \mathbb{E} a_n(t)a_m(t) = \mathbb{E} a_n^{**}a_m^{**} = \frac{1}{n+m-1}. \quad (1.8)$$

We also show that similar behaviors are observed with a different type of forcing. In particular, if the forcing is constant,

$$\dot{a}_n(t) = (n-1)a_{n-1}(t) - na_{n+1}(t) + \mathbf{1}_{n=1}, \quad (1.9)$$

then the system evolves to a unique steady state a_n^* which scales as

$$\lim_{t \rightarrow \infty} a_n(t) = a_n^* = \frac{\sqrt{\pi}\Gamma(\frac{n}{2})}{n\Gamma(\frac{n-1}{2})} \sim \sqrt{\frac{\pi}{n}}. \quad (1.10)$$

In summary, we see that the forced systems’ energy grows linearly with time if the energy is initially finite. Asymptotically, the system rearranges itself so that it reaches a state which dissipates energy at $t = \infty$. This state is chosen so that the dissipation rate matches the energy flux into the system from the forcing. Since the energy flux is finite, this leaves $|a_n| \approx 1/\sqrt{n}$ as the only choice. A slower decay rate would produce an infinite rate of dissipation and a faster decay rate would produce a system which conserved energy since the limit in (1.5) would be zero.

1.2 A second linear shell model

We introduce a second model which also exhibits interesting but different “blow-up” behavior. Consider

$$\dot{b}_n = (n-1)(n-\frac{1}{2})b_{n-1} - n(n+\frac{1}{2})b_{n+1} \quad (1.11)$$

for $n \in \mathbb{N}$ with the boundary condition $b_0(t) = 0$ for all t . As in the previous subsection, the unforced equation *formally* conserves energy since

$$\frac{1}{2} \frac{d}{dt} \sum_{n=1}^{\infty} b_n^2(t) = \sum_{n=1}^{\infty} [(n-1)(n-\frac{1}{2})b_{n-1}(t)b_n(t) - n(n+\frac{1}{2})b_{n+1}(t)b_n(t)] = 0. \quad (1.12)$$

This equality (as in the case of equation (1.1)) is only formal since, in general, the sum cannot be rearranged. As before, to gain insight we consider the partial sums. For $N \in \mathbb{N}$,

$$\frac{1}{2} \frac{d}{dt} \sum_{n \leq N} b_n^2(t) = -N(N + \frac{1}{2})b_N(t)b_{N+1}(t).$$

With this in mind, (1.11) will be called conservative if

$$\lim_{N \rightarrow \infty} N(N + \frac{1}{2})b_N(t)b_{N+1}(t) = 0,$$

and dissipative if

$$\liminf_{N \rightarrow \infty} N(N + \frac{1}{2})b_N(t)b_{N+1}(t) > 0.$$

Unlike the case of equation (1.1), if one assumes that $\lim_{N \rightarrow \infty} b_{N+1}/b_N$ exists and is in $(0, \infty)$, then the solution of equation (1.11) will be dissipative if $b_n \asymp 1/n$ as $n \rightarrow \infty$.

A solution satisfying $b_n(t) \asymp 1/n$ (if one exists) has finite energy: $\sum_{n=1}^{\infty} b_n^2(t) < \infty$. Thus, this model differs from the first example in that the system can dissipate energy even when the total energy is finite.

While we do not prove a general existence result as broad as for (1.1), we do show (in Theorem 9.1) that the dynamics for (1.11) are well defined if

$$\sum_{n=1}^{\infty} (-1)^n b_n(0) < \infty. \quad (1.13)$$

This is sufficient for our needs: In particular, it covers the case when $b_n \asymp 1/n$ as $n \rightarrow \infty$.

The differences between the first and second models are greater than simply the scaling. When started with initial conditions having finite energy the first model conserves energy. In fact the regularity at time t is the same as the regularity of the initial condition. In contrast, if we start (1.11) with initial data satisfying (1.13) (and hence with finite energy), the energy decays with time. Furthermore, for almost every $t > 0$ one has that $b_n(t) \asymp 1/n$ as $n \rightarrow \infty$ and there exists T , depending on the initial data, so that if $t > s > T$ then

$$\sum_{n=1}^{\infty} b_n^2(t) < \sum_{n=1}^{\infty} b_n^2(s) < \sum_{n=1}^{\infty} b_n^2(T) < \infty$$

and $\sum b_n^2(t) \rightarrow 0$ and $t \rightarrow \infty$. Turning to the forced setting, consider

$$\dot{b}_n = (n-1)(n-\frac{1}{2})b_{n-1} - n(n+\frac{1}{2})b_{n+1} + f(t)\mathbf{1}_{n=1}.$$

When $f(t) = \dot{W}(t)$ then

$$\mathbb{E}b_n^{**}b_{n+m}^{**} \asymp \frac{1}{n(n+m)} \quad \text{as } n \rightarrow \infty.$$

If $f(t) = 1$, we have

$$b_n^* \asymp \frac{1}{n} \quad \text{as } n \rightarrow \infty.$$

1.3 Inviscid limits of the first model

In practice, one is often interested in understanding the limit of equations when the explicit sources of dissipation are removed. To explore this question we investigate equation (1.1) with the addition of an overtly dissipative term and then study the limit as the dissipation is removed.

To understand our motivation, recall that we have seen that if the first model is started with finite energy initial data then the formal calculation presented in (1.2) is valid for all finite times as the energy must be infinity for (1.2) to fail. Yet, as time tends to infinity, the forced system converges to a steady state with infinite energy for which the calculation presented in (1.2) fails. In contrast, in the second model the analogous calculation, given in (1.12), fails at almost every positive time since $b(t) \asymp 1/n$ as $n \rightarrow \infty$ for almost every $t > 0$.

Since the coupling term produces dissipation at finite times in the second model, it is most interesting to study the effect of extra, explicit dissipation in the first model. To this end, we consider the stochastically forced version of the first model with extra, explicit dissipation sufficient to keep expected energy of the system finite for all times. In particular, the calculation in (1.2) is valid in the equilibrium state. We are interested in the structure of this invariant state and how it converges to the steady state without the explicit dissipation (as the dissipation is removed).

We will consider two cases, one for which the dissipative term is lower order than the coupling term and one for which it is higher order. Specifically, for $p \in \{0, 1\}$ and any $\nu > 0$, consider the equation

$$\dot{\alpha}_{n,\nu} = -2\nu(n-1)^p\alpha_{n,\nu} + [(n-1)\alpha_{n-1,\nu} - n\alpha_{n+1,\nu}] + \mathbf{1}_{n=1}\dot{W}(t).$$

The case $p = 0$ corresponds to the lower order damping and is analogous to what is called Eckman damping in the context of fluid mechanics. When $p = 1$ the perturbation is higher order than the coupling term and behaves as a viscous term in the language of fluid mechanics. As in the previous examples, we force the first coordinate with white noise. Assuming that $\sum |\alpha_{n,\nu}(0)|^2 < \infty$, it is straight forward to see that $\sum |\alpha_{n,\nu}(t)|^2$ stays finite and uniformly bounded in time for any $\nu > 0$. Hence, the system remains conservative for all time. Furthermore, as $t \rightarrow \infty$ the system converges to a random variable $\alpha_{n,\nu}^{**}$, whose distribution is the unique stationary measure for the system. Direct calculation in the spirit of (1.2) shows that

$$\mathbb{E} \sum_{n=1}^{\infty} (n-1)^p |\alpha_{n,\nu}^{**}|^2 = \frac{1}{\nu}.$$

Thus, there is no anomalous dissipation in the system: All of the dissipation which balances the forcing comes from the term $-2\nu(n-1)^p\alpha_n(t)$. In section 8, we will see that for $p \in \{0, 1\}$,

$$\mathbf{E}[\alpha_{n,\nu}^{**} - a_n^{**}]^2 \rightarrow 0 \quad \text{as } \nu \rightarrow 0.$$

Both of these steady states are Gaussian with mean zero. The way in which the variance of $\alpha_{n,\nu}^{**}$ converges (as $\nu \rightarrow 0$) to that of the $\nu = 0$ steady state a_n^{**} is different for the two values of p we consider. When $p = 0$ one has

$$\mathbf{E}[\alpha_{n,\nu}^{**}]^2 = 2^{1-2\nu}\Gamma(1+2\nu)\frac{\Gamma(2n+1-2\nu)}{\Gamma(2n+2)} \approx \frac{1}{n^{1+2\nu}}. \quad (1.14)$$

This shows that variances still decay like a power of n . Notice that for all $\nu > 0$ the total energy is finite. When $p = 1$ we do not obtain an exact formula but rather that

$$\frac{\kappa^2}{(\kappa+\nu)^{2n+2}} \frac{1}{2n+1} \leq \mathbf{E}[\alpha_{n,\nu}^{**}]^2 \leq \frac{1}{\kappa^{2n}} \frac{1}{2n+1} \quad (1.15)$$

where $\kappa^2 = 1 + \nu^2$. Since $\kappa > 1$ when $\nu > 0$, the $\alpha_{n,\nu}^{**}$ behave as the limiting a_n^{**} for small n but decay exponentially for large n .

1.4 Organization

The remainder of the paper is concerned with proving the statements made in this section. Section 2 contains a precise discussion of the concept of anomalous dissipation. In Section 3, we return to the first of the two models introduced in Section 1 and illustrate the range of possible dynamics by considering specific initial conditions for which the system can be explicitly solved. In Sections 4 and 5, we prove the existence of solutions and describe their properties for a wide range of initial data. In Section 7, we give the proofs of all of the preceding results. Section 6 discusses the forced setting for the first example and Section 8 discusses its inviscid limit. Finally, in Section 9 we discuss the second model introduced in Section 1: we first describe the qualitative behavior of solutions; then, we prove existence and uniqueness results both in the forced and unforced situations.

We note that some of the results about anomalous dissipation and the scaling of the solutions can also be obtained by formally taking the continuous limit in n of (1.1) and (1.11). In this limit, these equations formally reduce to hyperbolic conservation laws which were analyzed in [Sri05].

2 Preliminary: Definition of anomalous dissipation

The concepts of energy conservation, dissipation, and explosion are straightforward when the total energy of the system is finite. A system is *conservative* if the energy does not change with time. A system is *dissipative* (or *explosive*) if the total energy decreases (or increases) with time.

However, as the example in the previous section showed, it is possible to have solutions which one might call dissipative even though the total energy is infinite. We give a definition of the above terms which can be applied to situations where the total energy is infinite.

Given a time-dependent sequence, $\{a_n(t)\}_{n \in \mathbb{N}}$, define the energy in the block M to N , $M < N$, by

$$\mathcal{E}_{M,N}(t) = \sum_{n=M}^N |a_n(t)|^2.$$

A given block $\mathcal{E}_{M,N}$ is *dissipative* at time t if $\dot{\mathcal{E}}_{M,N}(t) < 0$. Similarly, we will say it is *explosive* if $\dot{\mathcal{E}}_{M,N}(t) > 0$. If $\dot{\mathcal{E}}_{M,N}(t) = 0$ then we say the block is *conservative* at time t . If $\dot{\mathcal{E}}_{M,N}(t) = 0$ for all M, N , and t , then the system is at a fixed point. (Note that this is consistent with Example 3.5 in the next section.)

We will say that the system is *locally dissipative* (*locally explosive*, or *locally conservative*) at time t if every finite block is *dissipative* (*explosive* or *conservative*) at time t .

In contrast, we will say that a system with $\mathcal{E}_{0,\infty} = \infty$ is *dissipative* at time t if the limit

$$\limsup_{N \rightarrow \infty} \frac{d}{dt} \mathcal{E}_{0,N}(t) < 0.$$

We say it is *explosive* at time t if

$$\liminf_{N \rightarrow \infty} \frac{d}{dt} \mathcal{E}_{0,N}(t) > 0.$$

When the limit exists we will refer to its absolute value as the *rate of energy dissipation* or the *rate of energy explosion* depending on the inequality which is satisfied. We say that the system with $\mathcal{E}_{0,\infty} = \infty$ is *conservative* at time t if

$$\lim_{N \rightarrow \infty} \frac{d}{dt} \mathcal{E}_{0,N}(t) = 0.$$

If we do not state time explicitly for any of these property, we mean that the properties holds for all finite times.

As the examples of the next section show, it is possible for $\lim_N \mathcal{E}_{0,N} = \infty$ while $\lim_N \dot{\mathcal{E}}_{0,N} = c < 0$.

Remark 2.1 It is important to notice that the above categorizations are not exhaustive. It is possible for a system not to fit into any of the categories. This is only an issue when the energy is infinite as we use the definitions at the start when the energy is finite.

3 The rich behavior of the first model

The system given by (1.1) possesses a number of interesting properties beyond those listed in the introductory section. In this section we explore the behavior through a number of examples. A relatively complete theory of the equation will be given in the two sections which follow. We begin with an existence result which covers all of the examples presented.

For an infinite vector $a = (a_1, a_2, \dots)$ with $a_i \in \mathbb{R}$, define $\rho(a)$ by

$$\frac{1}{\rho(a)} = \limsup_{n \rightarrow \infty} |a_n|^{1/n}. \quad (3.1)$$

($\rho(a)$ is simply the radius of convergence of the power series $\sum_n a_n z^n$). The following theorem gives an existence and uniqueness result for (1.1) sufficient for present needs.

In particular, it allows initial data with infinite energy ($\sum |a_n(0)|^2 = \infty$). A more complete description will be given in Theorem 4.1 of Section 4, where we describe the methodology for solving (1.1).

Theorem 3.1 *If $a(0) = (a_1(0), a_2(0), \dots)$ is an infinite vector of initial conditions such that $\rho(a(0)) > 0$, then there exists a unique solution $a(t)$ to (1.1) with initial conditions $a(0)$ which exists at least up to the time $t_* = \operatorname{arctanh}(\rho(a) \wedge 1)$.*

This existence result, whose proof is given in Section 4, covers a wide class of initial data. The dynamical behavior of (1.1) is quite rich. We list a number of exact solutions which display the range of possible behaviors. Explanations of how these results are obtained will be given in section 4.1. A general discussion of the qualitative properties of solutions of (1.1) will be given in Section 5.

Example 3.2 An energy conserving pulse heading out to infinity: Fixing $a_1(0) = 1$ and $a_n(0) = 0$ for all $n = 2, 3, \dots$ results in the dynamics

$$a_n(t) = \frac{\tanh^{n-1}(t)}{\cosh(t)}.$$

Even though the solution decays to zero pointwise in n as $t \rightarrow \infty$ it conserves energy: $\sum_{n=1}^{\infty} |a_n(t)|^2 = \sum_{n=1}^{\infty} |a_n(0)|^2 = 1$ for all $t \geq 0$. The fact that it conserves energy is consistent with the observations in equation (1.2) and (1.4) because

$$\frac{1}{2} \frac{d}{dt} \sum_{n \leq N} |a_n(t)|^2 = -N a_N(t) a_{N+1}(t) = -\frac{N \tanh^{N-1}(t) \tanh^N(t)}{\cosh^2(t)} \rightarrow 0$$

as $N \rightarrow \infty$. The dynamics of this solution can be understood as pulse moving out to larger and larger N with time while simultaneously spreading out. A simple calculation shows that $a_N(t)$ reaches its maximum at a time asymptotic to $\frac{1}{2} \log(1 + 4N)$ as $N \rightarrow \infty$. Hence, as $t \rightarrow \infty$ the n for which a_n which is cresting at time t scales as $\frac{1}{4} \exp(2t)$.

Example 3.3 Dissipative solution with finite dissipation rate: If

$$a_{n+1}(0) = \frac{(2n)!}{2^{2n}(n!)^2} \sim \frac{1}{\sqrt{\pi n}} \quad \text{for } n = 0, 1, \dots$$

then

$$a_{n+1}(t) = \frac{e^{-t/2}}{\sqrt{\cosh(t)}} \sum_{m=0}^n \frac{(2(n-m))!(2m)!}{2^{2n}((n-m)!m!)^2} \tanh^{n-m}(t) \quad \text{for } n = 0, 1, \dots$$

For each fixed n , we have

$$a_n(t) \sim \sqrt{2} e^{-t} \quad \text{as } t \rightarrow \infty,$$

so that the solution decays to zero pointwise in n as $t \rightarrow \infty$. On the other hand, since for any fixed time t

$$a_n(t) \sim \frac{e^{-t/2}}{\sqrt{\cosh t}} \frac{1}{\sqrt{\pi n}} \quad \text{as } n \rightarrow \infty,$$

it follows that $\sum_{n=1}^{\infty} |a_n(t)|^2 = +\infty$ for all time $t \geq 0$ and

$$\frac{1}{2} \frac{d}{dt} \sum_{n \leq N} |a_n(t)|^2 = -Na_N(t)a_{N+1}(t) \rightarrow -\frac{e^{-t}}{\pi \cosh t} < 0$$

as $N \rightarrow \infty$. For this solution, the calculation in (1.2) does not hold and the above inequality can be interpreted as a form of anomalous dissipation with finite dissipation rate.

Example 3.4 Dissipative solution with infinite dissipation rate: If $a_n(0) = 1$ for all $n = 1, 2, \dots$ then

$$a_n(t) = e^{-t} \quad \text{for all } n = 1, 2, \dots$$

This solution decays to zero pointwise in n as $t \rightarrow \infty$ and $\sum_{n=1}^{\infty} |a_n(t)|^2 = +\infty$ for all $t \geq 0$. Notice that

$$\frac{1}{2} \frac{d}{dt} \sum_{n \leq N} |a_n(t)|^2 = -Na_N(t)a_{N+1}(t) = -Ne^{-2t} \rightarrow -\infty$$

as $N \rightarrow \infty$. The formal calculation in (1.2) does not hold for this solution and, in terms of the definitions of Section 2, we view this as a form of anomalous dissipation with infinite dissipation rate.

Example 3.5 A fixed point: If $a_{2n}(0) = 0$ for $n = 1, 2, \dots$ and

$$a_{2n+1}(0) = \frac{(2n)!}{2^{2n}(n!)^2} \sim \frac{1}{\sqrt{\pi n}} \quad \text{for } n = 0, 1, \dots,$$

then

$$a_n(t) = a_n(0) \quad \text{for all } n = 1, 2, \dots$$

This solution is a fixed point of (1.1). Notice that

$$\frac{1}{2} \frac{d}{dt} \sum_{n=1}^N |a_n(t)|^2 = -Na_N(t)a_{N+1}(t) = 0$$

for all $N \in \mathbb{N}$, but $\sum_{n=1}^{\infty} |a_n(t)|^2 = +\infty$ since $a_{2n+1} \sim 1/\sqrt{\pi n}$ as $n \rightarrow \infty$.

Example 3.6 Explosive solution with infinite explosion time: If $a_n(0) = (-1)^{n+1}$ for $n = 1, 2, \dots$ then

$$a_n(t) = (-1)^{n+1} e^t \quad \text{for all } n = 1, 2, \dots$$

In this case, $\sum_{n=1}^{\infty} |a_n(t)|^2 = +\infty$ and

$$\frac{1}{2} \frac{d}{dt} \sum_{n=1}^N |a_n(t)|^2 = -Na_N(t)a_{N+1}(t) = Ne^{2t} \rightarrow +\infty$$

as $N \rightarrow \infty$. Thus, (1.2) does not hold for this solution and we see that the above limit is consistent with an infinite explosion rate for the solution. This is consistent with Theorem 3.1 as $\rho(a) = 1$ and $t_* = \operatorname{arctanh}(1) = \infty$.

Example 3.7 Explosive solution with finite explosion time: If $a_n(0) = (-1)^{n+1}\alpha^n$ with $\alpha > 1$ for $n = 1, 2, \dots$, then

$$a_n(t) = \frac{(-1)^{n+1}\alpha^n}{\cosh(t) - \alpha \sinh(t)} \quad \text{for } t < t_* = \operatorname{arctanh}(1/\alpha) \text{ and all } n = 1, 2, \dots$$

This solution blows up at $t = t_*$. In this case, $\sum_{n=1}^{\infty} |a_n(t)|^2 = +\infty$ for all $t < t_*$ and

$$\frac{1}{2} \frac{d}{dt} \sum_{n \leq N} |a_n(t)|^2 = -N a_N(t) a_{N+1}(t) = \frac{N \alpha^{2N+1}}{(\cosh(t) - \alpha \sinh(t))^2} \rightarrow +\infty$$

as $N \rightarrow \infty$ for all $t < t_*$. Notice that this example is consistent with Theorem 3.1 as $\rho(a) = 1/\alpha < 1$.

The above examples demonstrate the rich range of behavior of the model. In particular, some solutions grow coordinate-wise in time while others decay. The following result gives a criteria for the later.

Theorem 3.8 *Let $(a_1(0), a_2(0), \dots)$ be the infinite vector of initial conditions. If the limit*

$$\lim_{r \rightarrow -1^+} \sum_{n=1}^{\infty} a_n(0) r^n \tag{3.2}$$

exists and is finite, then for all $n \in \mathbb{N}$, $|a_n(t)| \rightarrow 0$ as $t \rightarrow \infty$.

Looking back at the examples, this result correctly separates those which decay to zero pointwise in n and those which do not.

Denote by ℓ^p the p -summable sequences: For $p > 0$,

$$\ell^p := \{(a_1, a_2, \dots) : \sum_{n=1}^{\infty} |a_n|^p < \infty\}. \tag{3.3}$$

If $a = (a_1, a_2, \dots) \in \ell^1$ then (3.2) exists and is finite. For future reference, we recall the norm $\|\cdot\|_{\ell^p}$ defined by $\|a\|_{\ell^p}^p \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} |a_n|^p$.

A complimentary question is to understand for which initial data the system conserves energy.

Theorem 3.9 *If $a(0) = (a_1(0), a_2(0), \dots) \in \ell^2$, then*

$$\|a(0)\|_{\ell^2}^2 = \|a(t)\|_{\ell^2}^2$$

for all time $t \geq 0$.

Example 3.2, 3.3 and 3.4 all have nice limits at $z = -1$ in the sense of (3.2) and they decay to zero as $t \rightarrow \infty$ as dictated by Theorem 3.8. It is particularly interesting to compare Example 3.4 and 3.6. Theorem 3.8 correctly says that the first decays to zero as time increases while declining to comment on the second.

Theorem 3.9 correctly states that Example 3.2 conserves energy. However, Theorem 3.9 is not completely satisfactory in that it only applies to solutions which have finite total energy. A number of our example have infinite energy. We now turn to understanding in detail the dynamics of (1.1).

4 Solution to the initial value problem

In this section we show that the initial value problem associated to equation (1.1) is well-posed and admits solutions for a wide class of initial data. Theorem 3.1 above is an immediate consequence of Theorem 4.1 below. After giving a general existence and uniqueness theorem, we present specific initial conditions (covered by the existence theorem) for which equation (1.1) admits solutions which conserve energy, dissipate energy, and blow up in finite time.

We begin by formally calculating a representation of the solution given by a generating function. We will verify that the representation is valid in the next section. Given initial conditions $\{a_n(0) : a_n(0) \in \mathbb{R}, n \in \mathbb{N}\}$, we assume that a solution $a_n(t)$ exists and define the generating function

$$G(z, t) = \sum_{n=0}^{\infty} a_{n+1}(t) z^n. \quad (4.1)$$

Proceeding formally, it is straight forward to verify that $G(z, t)$ would satisfy the following partial differential equation:

$$\frac{\partial G}{\partial t} = (z^2 - 1) \frac{\partial G}{\partial z} + zG \quad (4.2)$$

with initial condition

$$G_0(z) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} a_{n+1}(0) z^n. \quad (4.3)$$

The first term on the right hand side of (1.1) comes from $z \frac{\partial}{\partial z}(zG)$ and the second from $-\frac{\partial}{\partial z}G$.

One obtains an ansatz for the form of the solution by solving equation (4.2) by the method of characteristics. By verifying that this ansatz solves the equation, we obtain the following existence and uniqueness result whose proof is postponed until section 7.

Theorem 4.1 *Consider (1.1) with the initial condition $a_n(0)$ such that $G_0(z)$ is analytic in a neighborhood of the interval $(\alpha, 0]$. Then the solution of (1.1) exists and is unique for all $t \in [0, t_*]$ where $t_* = \operatorname{arctanh}(-\alpha \wedge 1)$.*

The unique solution is given by

$$a_n(t) = \oint_{\Gamma} \frac{G(z, t)}{2\pi i z^n} dz \quad (4.4)$$

with

$$G(z, t) \stackrel{\text{def}}{=} \frac{\psi_t(z)}{\cosh(t)} (G_0 \circ \phi_t)(z), \quad (4.5)$$

where

$$\psi_t(z) \stackrel{\text{def}}{=} \frac{1}{1 - z \tanh(t)} \quad \text{and} \quad \phi_t(z) \stackrel{\text{def}}{=} \frac{z - \tanh(t)}{1 - z \tanh(t)}. \quad (4.6)$$

Γ is any simple closed contour around the origin within the region of analyticity of $G(z, t)$ (which is non-empty).

It is worth noting that $G(z, t)$ solves the PDE given in (4.2) with $G(z, 0) = G_0(z)$ as initial condition.

Notice that finite time existence of solutions only requires that the initial data $a_n(0)$ have at most exponential growth in n , i.e. there exists $C > 0$ and $\gamma > 1$ such that for all $n \in \mathbb{N}$, $|a_n(0)| \leq C\gamma^n$. If the $a_n(0)$ decay exponentially then the solution exists for all times (i.e. $t_* = \infty$).

In addition, (4.4) implies that $a_{n+1}(t)$ is the n th term of the Taylor series expansion of $G(z, t)$ about $z = 0$. It is also straight forward to see that (4.4) defines a semigroup: for any suitable $f(z)$, let

$$(S_t f)(z) := \frac{1}{\cosh(t)} \psi_t(z) f(\phi_t(z)). \quad (4.7)$$

Then (4.5) can be expressed as $G(\cdot, t) = S_t G_0$ and it is easy to check that for any $t, s > 0$,

$$S_t \circ S_s f = S_s \circ S_t f = S_{t+s} f.$$

Since we are particularly interested in knowing the total energy of the solution, it is useful to notice that if $G(x, t)$ is as in (4.1) then

$$\|a(t)\|_{\ell^2}^2 \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} |a_n(t)|^2 = \frac{1}{2\pi} \int_0^{2\pi} |G(e^{i\theta}, t)|^2 d\theta \stackrel{\text{def}}{=} \|G(e^{i\theta}, t)\|_{L^2(S^1, d\theta)}^2.$$

We shall give more properties of the solutions of equation (1.1) in section 5 after a brief discussion of the examples given in the previous section.

4.1 Analysis of examples

We use Theorem 4.1 to calculate exact the solutions given in Section 3.

The initial data in Example 3.2, translates in to $G_0(z) = 1$, so that

$$G(z, t) = \frac{1}{\cosh(t) - z \sinh(t)}.$$

Calculating the Taylor series gives the quoted $a_n(t)$. In example 3.3 one obtains $G_0(z) = 1/\sqrt{1-z}$. Hence,

$$G(z, t) = \frac{e^{-t/2}}{\sqrt{\cosh(t)}} \frac{1}{\sqrt{(1-z)(1-z\tanh(t))}},$$

whose Taylor expansion produces the quoted $a_n(t)$. Example 3.4 yields $G_0(z) = 1/(1-z)$, $G(z, t) = e^{-t}/(1-z)$ and the desired a_n . Example 3.5 gives $G_0(z) = 1/\sqrt{1-z^2}$ and $G(z, t) = 1/\sqrt{1-z^2} \equiv G_0(z)$. Example 3.6 gives $G_0(z) = 1/(1+z)$ and $G(z, t) = \frac{e^t}{1+z}$. Example 3.7 gives $G_0(z) = \alpha/(1+\alpha z)$ and

$$G(z, t) = \frac{1}{\cosh(t) - \alpha \sinh(t)} \frac{\alpha}{1 + \alpha z} \quad \text{for } t < t_* = \operatorname{arctanh}(1/\alpha);$$

it blows up at $t = t_*$ as stated.

5 Properties of the solutions

We begin by presenting two results which are more quantitative versions of the results in Theorem 3.8 and Theorem 3.9. Together, they highlight the fact that it is possible to have a given coordinate converge to zero while no global energy dissipation is present in the system. This implies that there is a flux of energy out to higher and higher modes. It is also interesting that both of the next two results apply in some situations where the total energy is infinite.

Theorem 5.1 Suppose that $G_0(z)$ is analytic in a neighborhood of $(-1, 0]$ such that

$$G_0^+(-1) = \lim_{\substack{x \rightarrow -1^+ \\ x \in \mathbb{R}}} G_0(x) \quad (5.1)$$

exists and is finite. Then,

$$a_n(t) \sim 2e^{-t} G_0^+(-1) \quad \text{as } t \rightarrow \infty.$$

In particular, in a neighborhood of the origin $G(z, t) \sim 2e^{-t} G_0^+(-1)/(1 - z)$ as $t \rightarrow \infty$.

Remark 5.2 If $a_n = n^{-\alpha}$ for $\alpha < 1$, then $a_n(t) \sim 2e^{-t}\Gamma(1 - \alpha)$ as $t \rightarrow \infty$.

The next result contains Theorem 3.9 as well as giving control of higher Sobolev-like norms. We recall the Sobolev-like sequence spaces for $s \in \mathbb{R}$:

$$h_s \stackrel{\text{def}}{=} \{a = (a_1, a_2, \dots) : \|a\|_{h_s} < \infty\}$$

where the norm $\|\cdot\|_{h_s}$ is defined by $\|a\|_{h_s}^2 \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} n^{2s} |a_n|^2$.

Theorem 5.3 If

$$\|a(0)\|_{\ell^2}^2 \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} |a_n(0)|^2 < \infty$$

then

$$\|a(0)\|_{\ell^2}^2 = \|a(t)\|_{\ell^2}^2$$

for all $t \geq 0$. Similarly, for any $s \in \mathbb{N}$, if

$$\|a(0)\|_{h_s}^2 \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} n^{2s} |a_n(0)|^2 < \infty,$$

then for any $T < \infty$ there exists a constant, $C(T)$, such that

$$\sup_{t \in [0, T]} \|a(t)\|_{h_s}^2 \leq C(T).$$

5.1 Finer properties of solutions

We begin by giving conditions guaranteeing that the solution decays exponentially in time. We then turn to the case for which G_0 has singularities on the boundary of the unit circle. We close the section with a result which compares the dynamics obtained by placing a single unit of mass at different locations.

Let $D(r)$ denote the open disk of radius r about the origin. We begin with a simple criteria which guarantees that the a_n decay exponentially in n .

Theorem 5.4 Suppose $G_0(z)$ is analytic in the disk $D(1 + \eta)$ for some $\eta > 0$. Then, for each $T \geq 0$, there exist constants $\gamma, C > 0$ which depend only on η and T such that

$$\sup_{t \in [0, T]} |a_n(t)| \leq C\gamma^n \text{ for all } n \in \mathbb{N}.$$

In particular, the system conserves energy at all finite times.

In order to investigate the behavior of solutions when $G_0(z)$ is not analytic in $D(1 + \eta)$ for some $\eta > 0$, we introduce the following region:

$$\Delta(\zeta, \eta, \theta) \stackrel{\text{def}}{=} \{z : |z| \leq |\zeta| + \eta, |\arg(z - \zeta) - \arg(\zeta)| \geq \theta\}. \quad (5.2)$$

We begin with a careful analysis of the case when there is a single singularity on the unit circle. We contrast the cases for which the singularity is located at ± 1 in Theorem 5.5 and Theorem 5.7, respectively. The remaining cases are covered by Theorem 5.10.

Theorem 5.5 Assume that either

$$a_n(0) \sim Cn^{\alpha-1} \quad \text{as } n \rightarrow \infty$$

for some $C \neq 0$ or, more generally, that $G_0(z)$ satisfies both of the following conditions:

- i) There exist an $A \neq 0$ and $\alpha > 0$ so that

$$G_0(z) \sim A(1 - z)^{-\alpha} \quad \text{as } z \rightarrow 1. \quad (5.3)$$

- ii) $G_0(z)$ is analytic on $\Delta(1, \eta, \theta) \setminus \{1\}$ for some $\eta > 0$ and $0 < \theta < \pi/2$.

Then, for all time $t \geq 0$,

$$a_n(t) \sim \frac{A}{\Gamma(\alpha)} e^{(1-2\alpha)t} n^{\alpha-1} \quad \text{as } n \rightarrow \infty, \quad (5.4)$$

and the solution decays to zero as $t \rightarrow \infty$ pointwise in n ; more precisely,

$$a_n(t) \sim A2^{1-\alpha} e^{-t} \quad \text{as } t \rightarrow \infty.$$

Notice that in the setting of Theorem 5.5 the energy of the system is infinite for all $\alpha > 0$: $\sum_{n=1}^{\infty} a_n^2(t) = +\infty$. However, as a direct consequence of (5.4) in Theorem 5.5 and our definition of dissipation in section 2 the following is true:

Corollary 5.6 In the setting of Theorem 5.5: if $0 < \alpha < 1/2$ the system is conservative; if $\alpha = 1/2$ the system displays a finite dissipation rate; and, if $\alpha > 1/2$ the system displays an infinite dissipation rate.

We now consider a singularity at $z = -1$. The remaining points on the unit circle behave much like the $z = 1$ in that the solution decays to zero in time. They are discussed in Theorem 5.10 later in the section. The next result shows that if there is a singularity at $z = -1$ then the system can explode in time.

Theorem 5.7 Assume that either

$$a_n(0) \sim C(-1)^n n^{\alpha-1} \quad \text{as } n \rightarrow \infty$$

for some $C \neq 0$ or, more generally, that $G_0(z)$ satisfies both of the following conditions:

i) There exist an $A \neq 0$ and $\alpha > 0$ so that

$$G_0(z) \sim A(z+1)^{-\alpha} \quad \text{as } z \rightarrow -1. \quad (5.5)$$

ii) $G_0(z)$ is analytic on $\Delta(-1, \eta, \theta) \setminus \{-1\}$ for some $\eta > 0$ and $0 < \theta < \pi/2$.

Then, for all time $t \geq 0$,

$$a_n(t) \sim \frac{A}{\Gamma(\alpha)} e^{(2\alpha-1)t} (-1)^n n^{\alpha-1} \quad \text{as } n \rightarrow \infty$$

while for n fixed

$$a_n(t) \sim 2^{1-\alpha} e^{(2\alpha-1)t} A C_{n-1}^\alpha \quad \text{as } t \rightarrow \infty,$$

where C_n^α is the n^{th} coefficient of the Taylor series expansion at $z = 0$ of the function

$$\frac{1}{1-z} \left[\frac{1-z}{1+z} \right]^\alpha.$$

Notice that the energy of the system is again infinite for all $\alpha > 0$: $\sum_{n=1}^{\infty} |a_n(t)|^2 = +\infty$. However, we have:

Corollary 5.8 *In the setting of Theorem 5.7: If $0 < \alpha < 1/2$, the system is conservative; if $\alpha = 1/2$, it displays a finite explosion rate; and, if $\alpha > 1/2$, it displays an infinite explosion rate.*

Remark 5.9 One must be careful when interpreting the results in Corollary 5.8 since the system has infinite energy. For example, when $\alpha = 1/2$ then at each moment of time the system displays a finite explosion rate since energy is pumped in from infinity into any finite collection of modes. However, the rate must slow, falling to zero at $t = \infty$, since as $t \rightarrow \infty$ the system converges to the fixed point given in Example 3.5. This follows from the fact discussed in Section 4.1 that the fix point in Example 3.5 corresponds to the initial function $G(z) = B/\sqrt{1-z^2}$ for some $B \neq 0$.

We now give a more general result covering a singularity on the unit circle at any point other than -1 . Theorem 5.5 is a special case of the following result when $\zeta = 1$.

Theorem 5.10 *Let ζ be a point on the unit circle not equal to -1 . Assume that $G_0(z)$ behaves as*

$$G_0(z) \sim A(\zeta - z)^{-\alpha} \quad \text{with } A \neq 0 \text{ and } \alpha > 0 \text{ as } z \rightarrow \zeta, \quad (5.6)$$

and that $G_0(z)$ is analytic on $\Delta(\zeta, \eta, \theta) \setminus \{\zeta\}$ for some $\eta > 0$ and $0 < \theta < \pi/2$. Then, for all time $t \geq 0$,

$$a_n(t) \sim \frac{A}{\Gamma(\alpha)} \left[\left(\frac{1+\zeta}{2} \right) e^t + \left(\frac{1-\zeta}{2} \right) e^{-t} \right]^{1-2\alpha} \zeta_t^n n^{\alpha-1} \quad \text{as } n \rightarrow \infty$$

where $\zeta_t = \phi_t^{-1}(\zeta)$. (Notice that $\phi_t^{-1}(1) = 1$ and $\zeta_t \rightarrow 1$ as $t \rightarrow \infty$). The solution decays to zero as $t \rightarrow \infty$ pointwise in n :

$$a_n(t) \sim 2e^{-t} \frac{A}{(1+\zeta)^\alpha} \quad \text{as } t \rightarrow \infty.$$

From the examples above it is natural to conjecture that any singularity on the unit circle dominated by a polynomial-like singularity of degree less than 1/2 will not destroy energy conservation. It can be shown that this intuition is correct for a wide class of initial conditions. We already know that if the initial conditions have finite energy, then energy is conserved. By the reasoning in the section defining anomalous dissipation, it is enough to have

$$\lim_{n \rightarrow \infty} n^{\frac{1}{2}} |a_n| = 0. \quad (5.7)$$

This is possible even if the total energy is infinite. Using the Tauberian theorems in [Hil01] we can show that the dynamics preserves a subset of sequences satisfying (5.7). These solutions have infinite energy yet conserve energy in the sense of Section 2. As these results are tangential and a bit technical we do not give the details.

5.2 The fixed point

By combining the results of the previous section, one can understand a wide range of behavior. We illustrate this by examining the convergence to a fix point. In Example 3.5, we saw that the initial data corresponding to $G_0(z) = 1/\sqrt{1-z^2}$ was invariant under the dynamics. This function is analytic in the open unit disk and has two square root singularities on the unit circle: one at $z = 1$ and another at $z = -1$. Up to a technicality, we show that this is the only fixed point that characterizes the initial data that converges to it.

If the solution is to exist for all times the function $G_0(z)$ must be analytic in a neighborhood of $(-1, 0]$. From Theorem 5.1, we see that if the limit as $z \rightarrow -1$ of $G_0(z)$ is finite then the solution converges to zero pointwise in n as $t \rightarrow \infty$. Hence, if the system converges to a nontrivial steady state, it must have a singularity at $z = -1$. We assume that the singularity is power-like (i.e. $(z+1)^{-\alpha}$). One can likely deal with other singularities, however, we choose not to pursue this matter here. Theorem 5.7 implies that the singularity must be order 1/2 if it is a power; otherwise, the system would blow up or decay to zero. The question is: Can one make sense of the dynamics when $\sum |a_n|^2 = \infty$? The two facts above imply that any initial condition which has only polynomial singularities at $z = -1$ (if any) and which converges to a fixed point must be of the form:

$$G_0(z) = \frac{A}{(1+z)^{\frac{1}{2}}} + \tilde{G}_0(z) \quad (5.8)$$

where $A \neq 0$ and \tilde{G}_0 is analytic in a neighborhood of $(-1, 0]$ and $\tilde{G}_0(z) \rightarrow \tilde{G}_0^+ < \infty$ as $z \rightarrow -1$ from the right along the real axis. By Theorem 5.1 the dynamics starting from \tilde{G}_0 converge to zero as $t \rightarrow \infty$ pointwise in the sequence space. Theorem 5.5 says that the first term converges to $A\sqrt{2/(1-z^2)}$ as $t \rightarrow \infty$. Hence, all data of the form (5.8) converge to the fix point from Example 3.5. We have proved the following result:

Theorem 5.11 *Let $(a_1(0), a_2(0), \dots)$ be an initial condition such the solutions exists for all time and converges coordinate-wise to a fixed vector $(\bar{a}_1, \bar{a}_2, \dots)$ which is not the zero vector. Assuming that the $G_0(z)$ associated to this initial condition has only a power-like singularity at $z = -1$ then G_0 is of the form given in (5.8) and*

$$\sum_{n=0}^{\infty} \bar{a}_{n+1} z^n = A \sqrt{\frac{2}{1-z^2}}. \quad (5.9)$$

In particular, all fixed points of equation (1.1) are given by the Taylor series of (5.9).

Notice that the fixed points have infinite energy and have the property that all of the a_n , with n odd, are equal to zero. In Section 6, we will see that steady states with less a degenerate structure are obtained by forcing the system.

Remark 5.12 If one starts with an initial condition which has a polynomial singularity on the unit circle at $\zeta \neq -1$ and of an order $\beta + 1/2$ with $\beta > 0$, then for all finite times the norm $\|a(t)\|_{h_s}$ will be infinite if $s \geq -\beta$. Yet the dynamics still converges to the fix point given by Theorem 5.11. Hence, at $t = \infty$ all $\|\cdot\|_{h_s}$ norms are finite for $s < 0$.

5.3 The effect of shifting the initial condition

In Example 3.2 we described how a unit of mass placed at a_1 spreads out. The following theorem states that the solution obtained by placing a unit of mass in a_p behaves in the same way as the solution obtained by placing a mass at a_1 except that the picture is shifted p units down the chain.

Theorem 5.13 Let $G_0(z) = z^p$ for some $p \in \mathbb{N}$, then

$$a_n(t) = (-1)^p \frac{\tanh^{n+p}(t)}{\cosh(t)} + \beta_n(t) \frac{\tanh^{n-p+\xi}(t)}{\cosh(t)}$$

where $\xi = p - p \wedge n$ and $\beta_n(t)$ satisfies

$$|\beta_n(t)| \leq \begin{cases} n^p [1 - \tanh^{2p}(t)] & n \geq p \\ n^p \tanh^{p-n}(t) [1 - \tanh^{2n}(t)] & n < p \end{cases}$$

Proof Theorem 5.13. First observe that

$$G(z, t) = \frac{\cosh^{-1}(t)}{1 - z \tanh(t)} \left[\frac{z - \tanh(t)}{1 - z \tanh(t)} \right]^p.$$

Expanding this we find

$$G(z, t) = \frac{1}{\cosh(t)} \sum_{n=0}^{\infty} z^n [(-1)^p \tanh^{n+p}(t) + \beta_n \tanh^{n-p+\xi}(t)]$$

where

$$\beta_n(t) = \tanh^{p \wedge n}(t) \sum_{k=1}^{p \wedge n} [\tanh^{-1}(t) - \tanh(t)]^k [-\tanh(t)]^{p-k} \binom{p}{k} \binom{n}{k}.$$

So

$$|\beta_n(t)| \leq n^p \sum_{k=1}^{p \wedge n} [1 - \tanh^2(t)]^k [\tanh(t)]^{p \wedge n - k} [\tanh(t)]^{p-k} \binom{p}{k}.$$

In both cases the quoted estimate follows by using the binomial theorem. \square

6 The forced system

Since (1.1) may display anomalous dissipation, it is not unreasonable to expect that adding a forcing term to this equation may lead to a (statistical) steady state. We now show that this is indeed the case. Specifically, we study the system

$$\dot{a}_n(t) = (n-1)a_{n-1}(t) - na_{n+1}(t) + \mathbf{1}_{n=m}f(t), \quad (6.1)$$

where $m \in \mathbb{N}$, and $f(t)$ is either a constant forcing term, $f(t) = 1$, or a white-noise process, $f(t) = \dot{W}(t)$ (in the second case (6.1) has to be properly interpreted as an infinite system of coupled Itô stochastic differential equations).

As in the unforced setting, we represent the solution to (6.1) as in (4.4) for some $G(z, t)$. $a_n(t)$ is the n th coefficient in the Taylor series expansion of $G(z, t)$. By Duhamel's principle one sees that $G(z, t)$ must satisfy the generalization of (4.2) with the effect of $f(t)\mathbf{1}_{n=m}$ included:

$$\frac{\partial G}{\partial t} = (z^2 - 1)\frac{\partial G}{\partial z} + zG + F(z, t),$$

where $F(z, t) = z^{m-1}f(t)$ and the initial condition is $G(z, 0) = G_0(z)$. This equation is valid for both $f(t) = 1$ and $f(t) = \dot{W}(t)$ and forcing on any $m \in \mathbb{N}$. Using the semigroup representation defined in (4.7), the solution of the equation above can be represented as

$$G(z, t) = (S_t G_0)(z) + \int_0^t (S_{t-s}F)(z, s)ds. \quad (6.2)$$

We have

Theorem 6.1 Consider (6.1) with $m = 1$, $f(t) = 1$, and initial condition $a_n(0)$ satisfying the assumptions of Theorem 5.1. Then,

$$\lim_{t \rightarrow \infty} a_n(t) = a_n^* = \begin{cases} \sqrt{\frac{\pi}{2}} & \text{if } n = 1 \\ \frac{\sqrt{\pi}}{n-1} \cdot \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})} & \text{if } n \geq 2. \end{cases} \quad (6.3)$$

Remark 6.2 Notice that

$$a_n^* \sim \sqrt{\frac{\pi}{n}} \quad \text{as } n \rightarrow \infty$$

This implies that a_n^* has infinite energy: $\sum_{n=1}^{\infty} |a_n^*|^2 = +\infty$. This is consistent with the fact that the steady state must be dissipative to compensate for the effect of the forcing term since dissipative solutions must have infinite energy. In this simple example we can see how the forcing is balanced explicitly. Mirroring the calculation in (1.3) for any N : If we start the system $\{a_n^*\}$ at time $t = 0$ we have

$$\frac{1}{2} \frac{d}{dt} \sum_{n \leq N} |a_n^*(t)|^2 = a_1^* - Na_N^* a_{N+1}^* = 0$$

Hence, every block conserves energy as must happen at a fix point.

Theorem 6.3 Consider (6.1) with $m = 1$, $f(t) = \dot{W}(t)$, and initial condition $a_n(-T)$ satisfying the assumptions of Theorem 5.1. Then,

$$\lim_{T \rightarrow \infty} a_n(t) = a_n^{**}(t) \equiv \int_{-\infty}^t \frac{\tanh^{n-1}(t-s)}{\cosh(t-s)} dW(s) \quad a.s. \quad (6.4)$$

Remark 6.4 From (6.4), $a_n^{**}(t)$ is a Gaussian process with mean zero and covariance

$$\begin{aligned} \mathbb{E} a_n^{**}(t) a_m^{**}(t) &= \int_{-\infty}^t \frac{\tanh^{n+m-2}(t-s)}{\cosh^2(t-s)} ds \\ &= \frac{1}{n+m-1} \quad n, m \in \mathbb{N}. \end{aligned} \quad (6.5)$$

Again, this is consistent with the need for dissipation and implies that the invariant measure for (6.1) with a white-noise forcing is supported on functions with infinite energy (and, in particular, (6.5) is not trace-class).

In addition, notice that this is consistent with the fact that, at least in expectation, the steady state needs to dissipate precisely the energy pumped into the system. In fact, for any N ,

$$\frac{1}{2} \frac{d}{dt} \mathbb{E} \sum_{n \leq N} |a_n^{**}(t)|^2 = \frac{1}{2} - N \mathbb{E} a_N^{**}(t) a_{N+1}^{**}(t) = \frac{1}{2} - \frac{N}{2N} = 0.$$

Proof Theorem 6.1. The first term on the right hand-side accounts for the initial condition. Theorem 5.1 implies that $(S_t G_0) \rightarrow 0$ as $t \rightarrow \infty$. The second term is given explicitly by

$$\int_0^t (S_{t-s} F)(z, s) ds = \int_0^t \frac{\psi_{t-s}(z)}{\cosh(t-s)} (\phi_{t-s}(z))^{m-1} f(s) ds \quad (6.6)$$

Letting $f(t) = 1$ and $m = 1$, this expression becomes

$$\int_0^t (S_{t-s} F)(z, s) ds = \int_0^t \frac{\psi_{t-s}(z)}{\cosh(t-s)} ds.$$

It follows that

$$\lim_{t \rightarrow \infty} \int_0^t (S_{t-s} F)(z, s) ds = \frac{2}{\sqrt{1-z^2}} \left(\arctan \left(\frac{z-1}{\sqrt{1-z^2}} \right) + \pi \right).$$

a_n^* is the n th coefficient of the Taylor series expansion at $z = 0$ of this function. \square

Proof Theorem 6.3. Letting $f(t) = \dot{W}(t)$, $m = 1$, and considering the initial condition at $t = -T$, we have

$$\int_{-\infty}^t (S_{t-s} F)(z, s) ds = \int_{-\infty}^t \frac{\psi_{t-s}(z)}{\cosh(t-s)} dW(s).$$

$a_n^{**}(t)$ is the n th coefficient of the Taylor series expansion at $z = 0$ of this function.

\square

7 Proofs of the main theorems

We begin by making a number of observations which will be used in the proofs. ψ_t and ϕ_t each have a single simple pole at $z = 1/\tanh(t)$. Hence, at any finite time both are analytic in an open disk containing the closed unit disk and the Taylor coefficients of their expansions about zero converge to zero exponentially in n .

For any fixed $t > 0$ the map ϕ_t is a fractional linear transformation which bijectively maps the open unit disk onto itself and leaves the unit circle invariant. The points $z = 1$ and $z = -1$ are the two fix points. In addition, for every fixed $z \in D(1) \setminus \{1\}$ and fixed neighborhood \mathcal{N} of -1 , there exists a time $T(z, \mathcal{N})$ such that $\phi_t(z) \in \mathcal{N} \cap D(1)$ for all $t > T(z, \mathcal{N})$. The behavior of $G_0 \circ \phi_t$ in a neighborhood of the origin will be important in the analysis which follows. Observe that $\phi_t(0) = -\tanh(t)$ and for sufficiently small $r > 0$, $\{\rho e^{i\theta} : \theta \in [0, 2\pi], \rho \in [0, r]\}$ is mapped approximately to $\{-\tanh(t) + \rho(1 - \tanh^2(t))e^{i\theta} : \theta \in [0, 2\pi], \rho \in [0, r]\}$ and strictly into the closed disk

$$\mathcal{E}_r = \left\{ -\tanh(t) + \rho(1 - \tanh^2(t))e^{i\theta} : \theta \in [0, 2\pi], \rho \in [0, r/(1-r)] \right\}.$$

For sufficiently small $r > 0$, \mathcal{E}_r is strictly contained in the unit disk for all times t . Furthermore, \mathcal{E}_r is bounded away from the boundary by two lines emanating from -1 of the form $\{-1 + \rho e^{\pm i\theta} : \rho \geq 0\}$ for some fixed $\theta \in (0, \pi/2)$.

We recall a basic fact from complex analysis which will be used repeatedly in the arguments that follow. To show that the Taylor coefficients about zero of $G_0 \circ \phi_t$ converges to those of F as $t \rightarrow \infty$ it is sufficient that for all $t \geq 0$, F and $G_0 \circ \phi_t$ are analytic in a fixed, t independent neighborhood of the origin and that $G_0 \circ \phi_t$ converges uniformly to F on that neighborhood.

Proof Theorem 4.1. In order that equation (4.4) be well defined, $G(z, t)$ needs to be analytic in a neighborhood of $z = 0$. Since ψ_t is analytic for all $z \in D(1)$ and each $t > 0$, the analyticity of $G(z, t)$ about $z = 0$ is equivalent to the analyticity of $G_0(z)$ about $\phi_t(0) = -\tanh(t)$. As t increases, $\phi_t(0)$ decreases monotonically along the real axis from 0 to -1 . We need only show that G_0 is analytic in an open neighborhood of $[-\tanh(t), 0]$ in order to complete the proof that equation (4.4) is well defined for all $t < t_*$. G_0 is analytic in a neighborhood of the closed interval $[-\tanh(t), 0]$ since $[-\tanh(t), 0] \subset (\alpha, 0]$. Appealing to the arguments stated at the beginning of this section we see that the image of a small ball about the origin under the mapping ϕ_t lies in a thin strip about $[-\tanh(t), 0]$. Hence, the reconstruction formula of equation (4.4) is well defined because Γ can be deformed to lie in a sufficiently small ball about the origin.

If $t_* < \infty$, then $G(z, \tanh(t_*))$ fails to be analytic at $z = 0$ since $G_0(z)$ is not analytic at $\alpha = -\tanh(t_*)$; we cannot continue the solution in this case.

To see that the $a_n(t)$ (defined as in the statement of the theorem) do define a solution, observe that by the definition of $G(z, t)$ and integration by parts

$$\begin{aligned} \dot{a}_n(t) &= \oint_{\Gamma} \frac{\partial_t G(z, t)}{2\pi i z^n} dz = \oint_{\Gamma} \frac{(z^2 - 1)\partial_z G(z, t) + zG(z, t)}{2\pi i z^n} dz \\ &= (n-1) \oint_{\Gamma} \frac{G(z, t)}{2\pi i z^{n-1}} dz - n \oint_{\Gamma} \frac{G(z, t)}{2\pi i z^{n+1}} dz = (n-1)a_{n-1}(t) - na_{n+1}(t). \end{aligned}$$

This shows that $a_n(t)$ given by (4.4) is indeed a solution of (1.1) for the initial condition $a_n(0)$ as long as $G(z, t)$ is analytic around $z = 0$.

Finally, to show that the $a_n(t)$ defined by equation (4.4) is the unique solution of (1.1) for the initial condition $a_n(0)$, note that if two different solutions exist for the same initial condition, then their associated $G(z, t)$ must both satisfy (4.2) for the same initial condition $G_0(z)$. Since the solution of (4.2) is unique, this leads to a contradiction. \square

Proof Theorem 5.1. Appealing to the discussion at the beginning of the section and the fact that $\cosh(t) \sim \frac{1}{2}e^t$ as $t \rightarrow \infty$, it is enough to show that $\psi_t(z)(G_0 \circ \phi_t)(z)$ converges uniformly to $G^+(-1)/(1 - z)$ on some neighborhood of the origin. First, observe that $\psi_t(z)$ converges to $1/(1 - z)$ as $t \rightarrow \infty$ uniformly on any disk contained within the unit disk.

From the discussion at the beginning of the section we see that for all $t > 0$ the disk of radius $1/10$ is mapped to a disk contained entirely in the open unit disk and bounded away from the unit circle by lines emanating from -1 of a constant angle. Hence, we have

$$\lim_{t \rightarrow \infty} \sup_{|z| < 1/10} |G_0(\phi_t(z)) - G_0^+(-1)| = 0 .$$

 \square

Proof Theorem 5.3. Since $(a_1(0), a_2(0), \dots)$ is square summable, $g(r, \theta) := G_0(re^{i\theta})$ is in $L^2(d\theta)$ of the unit circle for all $r \in [0, 1]$. Hence, by Plancherel's theorem,

$$\sum_{n=1}^{\infty} |a_n(t)|^2 = \frac{1}{2\pi} \int_0^{2\pi} |G(e^{i\theta}, t)|^2 d\theta = \frac{1}{2\pi} \int_0^{2\pi} \frac{|\psi_t(e^{i\theta})|^2}{\cosh^2(t)} |G_0(\phi_t(e^{i\theta}))|^2 d\theta$$

Introducing the change of variable $e^{i\eta} = \phi_t(e^{i\theta})$, we see that

$$\begin{aligned} \sum_{n=1}^{\infty} |a_n(t)|^2 &= \frac{1}{2\pi} \int_0^{2\pi} \frac{|\psi_t(\phi_t^{-1}(e^{i\eta}))|^2}{\cosh^2(t)} |G_0(e^{i\eta})|^2 \frac{1}{|\phi_t(\phi_t^{-1}(e^{i\eta}))|} d\eta \\ &= \frac{1}{2\pi} \int_0^{2\pi} |G_0(e^{i\eta})|^2 d\eta = \sum_{n=1}^{\infty} |a_n(0)|^2 . \end{aligned}$$

To obtain the bounds on the weighted norms, we notice that for $s \in \mathbb{N}$

$$\sum_{n=1}^{\infty} n^{2s} |a_n(t)|^2 = \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\partial^s G}{\partial \theta^s}(e^{i\theta}, t) \right|^2 d\theta .$$

Using the same change of variable one easily shows that this term can be bounded in terms of the L^2 norms of $\frac{\partial^s}{\partial \theta^s} G_0(e^{i\theta})$ for $r \leq s$. By assumption, these norms are finite. \square

Proof Theorem 5.4. Fix $T > 0$. By the considerations at the beginning of the section, one sees that for $t \in [0, T]$ the map $\psi_t(z)(G_0 \circ \phi_t)(z)$ remains analytic in $D(1 + \eta_1)$ for a sufficiently small η_1 depending only on T and η . More precisely, η_1 is picked to ensure that $D(1 + \eta_1)$ maps into $D(1 + \eta)$ for all $t \in [0, T]$; the only remaining constraint on η_1 is that $1 + \eta_1 < \frac{1}{\tanh T}$ so that $\psi_t(z)$ is also analytic in $D(1 + \eta_1)$. Hence, the power series converges absolutely on $D(1 + \eta_1/2)$ and $|a_n(t)| \leq C(1 + \eta_1/2)^n$ for all $n \in \mathbb{N}$ for some $C > 0$. \square

Proof Theorem 5.5, Theorem 5.10 and Corollary 5.6. Theorem 5.5 is a special case of Theorem 5.10 so we concentrate on the later. By the discussion in the proof of Theorem 5.1, it is clear that for each moment of time $t > 0$ there exists a $\eta_1 > 0$ so that $G(z, t)$ is analytic on $\Delta(\zeta_t, \eta_1, \theta_1) \setminus \{\zeta_t\}$. η_1 may be chosen to be sufficiently small in order to avoid other singularities of G_0 which initially lie outside $\Delta(\zeta, \eta, \theta) \setminus \{\zeta\}$ and approach $D(1)$ under the dynamics of ϕ_t . A similar consideration needs to be taken into account for θ and may result in an increase of θ to a new θ_1 . As $z \rightarrow \zeta_t$, we see that

$$\begin{aligned} G(z, t) &\sim \frac{[\cosh(t) - \zeta_t \sinh(t)]^{\alpha-1}}{[\cosh(t) + \zeta \sinh(t)]^\alpha} \frac{A}{(\zeta_t - z)^\alpha} \\ &= \left[\left(\frac{1+\zeta}{2} \right) e^t + \left(\frac{1-\zeta}{2} \right) e^{-t} \right]^{1-2\alpha} \frac{A}{(\zeta_t - z)^\alpha}. \end{aligned}$$

The result on the asymptotics in n then follows from Theorem A.1 in the appendix since $G(z, t)$ is analytic on $\Delta(\zeta_t, \eta_1, \theta_1) \setminus \{\zeta_t\}$. The result for fixed n as $t \rightarrow \infty$ is just a restatement of Theorem 5.1 in this context. The Corollary follows directly from the discussion in section 2. \square

Proof Theorem 5.7 and Corollary 5.8. The proof is similar to that of Theorem 5.5. Since $z = -1$ is a fixed point for ϕ_t for all $t > 0$, $G(z, t)$ has a singularity at $z = -1$ inherited from $G_0(z)$. Since the circle is invariant under ϕ_t , for sufficiently small $\eta_1 > 0$ and θ_1 sufficiently close to $\pi/2$ we have that $G(z, t)$ is analytic on $\Delta(-1, \eta_1, \theta_1)$. Direct calculation yields:

$$G(z, t) \sim \frac{1}{\cosh(t) + \sinh(t)} \left[\frac{1 + \tanh(t)}{1 - \tanh(t)} \right]^\alpha \frac{A}{(1+z)^\alpha} = \frac{Ae^{(2\alpha-1)t}}{(1+z)^\alpha} \quad \text{as } z \rightarrow -1.$$

We obtain the quoted result by applying Theorem A.1 from the appendix.

The asymptotics in time follow from the fact that

$$G(z, t) \sim \frac{Ae^{(2\alpha-1)t}}{1-z} \left(\frac{1-z}{1+z} \right)^\alpha \quad \text{as } t \rightarrow \infty$$

and direct expression of the right hand side in a Taylor series in z . Corollary 5.8 follows from the discussion on anomalous dissipation in section 2 and the above results. \square

8 Inviscid limits

We return to the analysis of the inviscid limits of (1.1). Fixing $p \in \mathbb{N}$ and defining

$$\Lambda_n = \prod_{k=1}^p (n-k)$$

with the convention that $\Lambda_n = 1$ if $p = 0$, we consider

$$\dot{\alpha}_{n,\nu} = -2\nu\Lambda_n\alpha_{n,\nu} + [(n-1)\alpha_{n-1,\nu} - n\alpha_{n+1,\nu}] + \mathbf{1}_{n=1}\dot{W}(t). \quad (8.1)$$

As mentioned in Section 1.1, it is straightforward to see that this system converges to a random variable $\alpha_\nu^{**} = (\alpha_{1,\nu}^{**}, \alpha_{2,\nu}^{**}, \dots)$. In fact, one has

$$\mathbb{E} \sum_n \Lambda_n |\alpha_{n,\nu}^{**}|^2 = \frac{1}{\nu}.$$

Thus, the system does not display anomalous dissipation; the dissipation which balances the energy injection (due to the forcing) comes from the term $-2\nu n^p \alpha_n(t)$.

Setting

$$\mathcal{G}_\nu(z, t) = \sum_{n=0}^{\infty} \alpha_{n+1, \nu}(t) z^n,$$

one sees that

$$\frac{\partial \mathcal{G}_\nu}{\partial t} = (z^2 - 1) \frac{\partial \mathcal{G}_\nu}{\partial z} + z \mathcal{G}_\nu - 2\nu z^p \frac{\partial^p \mathcal{G}_\nu}{\partial z^p} + \dot{W}(t). \quad (8.2)$$

Using the variation of constants formula we obtain

$$\mathcal{G}_\nu(\cdot, t) = \mathcal{S}_{t, \nu} \mathcal{G}_\nu(\cdot, 0) + \int_0^t (\mathcal{S}_{t-s, \nu} 1) dW_s.$$

We will concentrate on the case $p \in \{0, 1\}$. By the method of characteristics, we find that

$$(\mathcal{S}_{t, \nu} f)(z) = \begin{cases} \frac{e^{-2\nu t}}{\cosh(t)} \psi_t(z) f(\phi_t(z)) & \text{for } p = 0 \\ \frac{1}{\cosh(\kappa t)} \psi_{\kappa t}(z - \nu) f(\nu + \kappa \phi_{\kappa t}(z - \nu)) & \text{for } p = 1 \end{cases}$$

where $\kappa^2 = 1 + \nu^2$.

It is interesting to contrast the regularizing effect of the different terms. When $p = 0$, $\mathcal{S}_{t, \nu}$ simply dissipates energy at a faster rate than S_t . When $p = 1$, $\mathcal{S}_{t, \nu}$ has a stronger regularizing effect than $\mathcal{S}_{t, \nu}$, in that the characteristics are attracted to the circle $(\nu - \kappa)e^{i\theta}$ inside of the unit disk and the singularity of $\psi_{\kappa t}(z - \nu)$ stays uniformly bounded outside of the unit disk for all times. Hence if f has a radius of convergence greater than $\nu - \kappa \sim -1 + \nu$ then $S_t^{\nu, 1} f$ is analytic on a disk with radius greater than one all times uniformly.

For fixed t , $\mathcal{S}_{t, \nu} f$ converges to S_t as $\nu \rightarrow 0$ uniformly on a neighborhood of the origin. Since one also has that $S_t f$, $\mathcal{S}_{t, \nu} f$ all go to zero uniformly on the open disk as $t \rightarrow \infty$ for f bounded on the unit disk, we have that $\int_{-\infty}^t (\mathcal{S}_{t-s, \nu} 1) dW_s$ converges to $\int_{-\infty}^t (S_{t-s} 1) dW_s$ in mean squared as $\nu \rightarrow 0$. As before we are primarily interested in these solutions. In this setting they are given by:

$$\alpha_{n, \nu}^{**}(t) = \begin{cases} \int_{-\infty}^t e^{-2\nu(t-s)} \frac{\tanh(t-s)^{n-1}}{\cosh(t-s)} dW(s) & p = 0 \\ \int_{-\infty}^t \frac{\kappa}{[\kappa + \nu \tanh(\kappa(t-s))]^n} \frac{\tanh(\kappa(t-s))^{n-1}}{\cosh(\kappa(t-s))} dW(s) & p = 1 \end{cases} \quad (8.3)$$

Theorem 8.1 *For all $n \in \mathbb{N}$, $t \in \mathbb{R}$, and $p = 0, 1$, one has*

$$\lim_{\nu \rightarrow 0} \mathbb{E}[\alpha_{n, \nu}^{**}(t) - a_n^{**}(t)]^2 = 0$$

Hence $\alpha_{n, \nu}^{**}(t)$ converges to $a_n^{**}(t)$ as $\nu \rightarrow 0$ for $p = 0, 1$. Furthermore, one has the estimates given in (1.14) and (1.15).

Proof Theorem 8.1. Fix any t . Consider $\alpha_n(t, \nu, 0)$ and $a_n(t)$ starting from initial condition zero at time T with $T < t$. As $T \rightarrow -\infty$, $\alpha_n(t, \nu, 0)$ and $a_n(t)$ converge respectively to $\alpha_n^{**}(t, \nu, 0)$ and $a_n^{**}(t)$.

By the same argument as Theorem 6.3, one see that (8.3) holds. Subtracting (8.3) from (6.4), one obtains

$$\begin{aligned}\mathbb{E}[\alpha_n^{**}(t, \nu, 0) - a_n^{**}(t)]^2 &= 2 \int_0^\infty [(1+u)^{-\nu} - 1]^2 \frac{u^{2(n-1)}}{(u+2)^{2n}} du \\ &\leq 2 \int_0^\infty [(1+u)^{-\nu} - 1]^2 \frac{1}{(u+1)^2} du = \frac{4\nu^2}{(1+\nu)(1+2\nu)},\end{aligned}$$

which implies that $\alpha_n^{**}(t, \nu, 0) \rightarrow a_n^{**}(t)$ almost surely as $\nu \rightarrow 0$. The convergence of in the other cases is similar. Applying the Itô isometry to (8.3) proves the quoted value of $\mathbb{E}[\alpha_n^{**}(t, \nu, p)]^2$ for $p = 0$. The other estimates follow from

$$\frac{1}{\kappa + \nu} \leq \frac{1}{\kappa + \nu \tanh(t)} \leq \frac{1}{\kappa}$$

which holds for $t \geq 0$. \square

Remark 8.2 At first glance, it might seem more natural to consider the system

$$\dot{\tilde{\alpha}}_{n,\nu} = -2\nu n^p \tilde{\alpha}_{n,\nu} + [(n-1)\tilde{\alpha}_{n-1,\nu} - n\tilde{\alpha}_{n+1}] + \mathbf{1}_{n=1} \dot{W}(t).$$

This leads to the following equation for the generating function $\tilde{\mathcal{G}}_\nu$:

$$\frac{\partial \tilde{\mathcal{G}}_\nu}{\partial t} = (z^2 - 1) \frac{\partial \tilde{\mathcal{G}}_\nu}{\partial z} + z \tilde{\mathcal{G}}_\nu - 2\nu z^p \frac{\partial^p \tilde{\mathcal{G}}_\nu}{\partial z^p} - 2\nu \mathcal{D}^p + \dot{W}(t).$$

where \mathcal{D}^p is p applications of the operator defined by $(\mathcal{D}f)(z) = \frac{\partial}{\partial z}(zf(z))$ and \mathcal{D}^0 is the identity operator. Hence, we see that the extra dissipative term contains derivative of all orders less then or equal to p . Not surprisingly, the result is a mixture of the behavior of (8.1) for all orders less than or equal to p . In particular, when $p = 1$ the asymptotic (in time) behavior is given by

$$\tilde{\alpha}_{n,\nu}^{**}(t) = \int_{-\infty}^t \frac{\kappa e^{-\nu(t-s)}}{[\kappa + \nu \tanh(\kappa(t-s))]^n} \frac{\tanh(\kappa(t-s))^{n-1}}{\cosh(\kappa(t-s))} dW(s)$$

and satisfying the following estimate:

$$\frac{\kappa^2}{(\kappa + \nu)^{2n+2}} \mathbb{E}[\alpha_{n,\nu\kappa/2}^{**}]^2 \leq \mathbb{E}[\tilde{\alpha}_{n,\nu}^{**}]^2 \leq \frac{1}{\kappa^{2n}} \mathbb{E}[\alpha_{n,\nu\kappa/2}^{**}]^2.$$

9 A second linear shell model

We begin the analysis of the second model (1.11) by giving general conditions for the existence of a unique solution of the initial value problem. The technique is similar to that used in Section 4.

Theorem 9.1 *Let $\{b_n(0)\}$ be such that*

$$\sum_{n=1}^{\infty} (-1)^n b_n(0) < \infty. \quad (9.1)$$

Then the solution of (1.11) exists and is unique for all positive times. It can be represented as

$$b_n(t) = \frac{(-1)^{n+1}}{(2n-1)!} \frac{\partial^{2n-1} H}{\partial x^{2n-1}}(0, t) \quad (9.2)$$

where

$$H(x, t) = \mathbb{E}_x H_0(X(t)) \exp \left(-\frac{1}{2} \int_0^t X^2(s) ds \right), \quad (9.3)$$

and $X(t)$ satisfies the stochastic differential equation

$$dX(t) = -X^3(t)dt + \frac{1}{\sqrt{2}} \sqrt{1 - X^4(t)} dW(t). \quad (9.4)$$

\mathbb{E}_x denotes the expectation conditional on $X(0) = x \in [-1, 1]$ and

$$H_0(x) = \sum_{n \in \mathbb{N}} (-1)^{n+1} b_n(0) x^{2n-1}. \quad (9.5)$$

Remark 9.2 Alternatively, $H(x, t)$ can be expressed as

$$H(x, t) = \sum_{n \in \mathbb{N}} (-1)^{n+1} b_n(t) x^{2n-1}. \quad (9.6)$$

where $b_n(t)$ solves (1.11).

Remark 9.3 If the sequence $\{b_n(0)\}$ is monotone and converges to zero as $n \rightarrow \infty$ then the condition in (9.1) holds.

The following theorem summarizes the most interesting properties of solutions of (1.11).

Theorem 9.4 Suppose that $b_n(0)$ satisfies (9.1). Then, for any positive time $t > 0$,

$$\lim_{n \rightarrow \infty} (2n+1)b_{2n+1}(t) = \bar{C}_1(t) \text{ and } \lim_{n \rightarrow \infty} 2nb_{2n}(t) = \bar{C}_2(t), \quad (9.7)$$

where $\bar{C}_1(t), \bar{C}_2(t) \in \mathbb{R}$, $\bar{C}_1(t), \bar{C}_2(t) \neq 0$ for all but finitely many $t \in [0, \infty)$. In particular, there exists a $T > 0$ such that for all $t \geq T$, the solution of (1.11) is dissipative and satisfies

$$\sum_{n=1}^{\infty} b_n^2(t) < \sum_{n=1}^{\infty} b_n^2(T) < \infty. \quad (9.8)$$

In fact, $\sum_{n=1}^{\infty} b_n^2(t) \rightarrow 0$ as $t \rightarrow \infty$.

Remark 9.5 The fact that equation (1.11) dissipates energy at finite times is implicit in the representation (9.3). As time grows, the factor $\exp(-\frac{1}{2} \int_0^t X^2(s) ds)$ converges to zero as $\exp(-ct)$ almost surely for some positive deterministic c . (This follows from the law of large numbers and the verifiable assumption that the process is ergodic.) Hence, $H(x, t)$ converges to zero uniformly in x as $t \rightarrow \infty$.

Writing (9.3) as $H(x, t) = (T_t H_0)(x)$, it is easy to see that T_t defines a (Feller) semigroup with generator L defined by

$$\begin{aligned} (Lf)(x) &= \frac{(1-x^4)}{4} \frac{\partial^2 f}{\partial x^2} - x^3 \frac{\partial f}{\partial x} - \frac{x^2}{2} f \\ &= \frac{1}{4} \frac{\partial}{\partial x} \left((1-x^4) \frac{\partial f}{\partial x} \right) - \frac{x^2}{2} f. \end{aligned} \quad (9.9)$$

for $f \in C^2([-1, 1])$. In addition, $H(x, t)$ satisfies

$$\frac{\partial H}{\partial t} = \frac{1}{4} \frac{\partial}{\partial x} \left((1-x^4) \frac{\partial H}{\partial x} \right) - \frac{x^2}{2} H, \quad (9.10)$$

with initial condition $H(x, 0) = H_0(x)$ for $x \in [-1, 1]$. One can check that the boundaries at $x = \pm 1$ are entrance boundaries for (9.10) and $H(x, t)$ satisfies

$$\lim_{x \rightarrow \pm 1} (1-x^4) \frac{\partial H}{\partial x} = 0. \quad (9.11)$$

Proof Theorem 9.1.: Noting that

$$Lx^{2n-1} = -n(n + \frac{1}{2})x^{2n+1} + (n-1)(n - \frac{1}{2})x^{2n-3},$$

we compute

$$\begin{aligned} &\sum_{n=1}^{\infty} (-1)^{n+1} b_n(t) x^{2n-1} \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} \left((n-1)(n - \frac{1}{2})b_{n-1}(t) - n(n + \frac{1}{2})b_{n+1}(t) \right) x^{2n-1} \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} b_n(t) \left(-n(n + \frac{1}{2})x^{2n+1} + (n-1)(n - \frac{1}{2})x^{2n-3} \right) \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} b_n(t) L(x^{2n-1}). \end{aligned}$$

□

Proof Theorem 9.4.: Associated with (9.10) we have the eigenvalue problem

$$-\lambda \phi = \frac{1}{4} \frac{d}{dx} \left((1-x^4) \frac{d\phi}{dx} \right) - \frac{x^2}{2} \phi, \quad (9.12)$$

subject to the boundary conditions

$$\lim_{x \rightarrow \pm 1} (1-x^4) \frac{\partial \phi}{\partial x} = 0.$$

It is straightforward to see that the operator in (9.12) equipped with the boundary condition in (9.11) is self-adjoint in $L^2[-1, 1]$. We now explain why this operator has discrete spectrum. A standard calculation shows that the boundary is an “entrance

boundary” in the sense of Feller ([Fel54, McK56]), i.e. the diffusion (9.4), if started from the boundary, enters $(-1, 1)$ and does not return to the boundary.

Define $L^V = \frac{1}{4}\partial_x(1-x^4)\partial_x + V(x)$ where $V(x) = -\frac{x^2}{2}$. By standard PDE theory,

$$\partial_t u = L^V u,$$

subject to the condition $\lim_{t \rightarrow 0} u(t, x) = \delta_y(x)$, has smooth solution in $(-1, 1)$ for any $t > 0$. We denote this solution by $p_t^V(x, y)$. For a fixed $t > 0$, $p_t^V(x, y)$ is Lipschitz for $x \in [-1 + \epsilon, 1 - \epsilon]$ and $y \in [-1, 1]$ with a fixed Lipschitz constant C_t^ϵ and $\sup_{x,y \in [-1,1]} |p_t^V(x, y)| < D_t$. Consider the solution to the following initial value problem: Let $f \in L^2([-1, 1])$ and solve

$$\partial_t u = L^V u \text{ with } u(0, x) = f(x).$$

The solution is given by

$$u(t, x) = \int p_t^V(x, y)f(y)dy.$$

$u(t, x)$ is Lipschitz for $x \in [-1 + \epsilon, 1 - \epsilon]$ as the following simple estimate shows.

$$\begin{aligned} |u(t, x) - u(t, x')| &= \left| \int_{-1}^1 (p_t^V(x, y) - p_t^V(x', y))f(y)dy \right| \leq \int_{-1}^1 |p_t^V(x, y) - p_t^V(x', y)| |f(y)| dy \\ &\leq C_t^\epsilon |x - x'| \int_{-1}^1 |f(y)| dy \leq 2C_t^\epsilon |x - x'| \left(\int_{-1}^1 |f(y)|^2 dy \right)^{\frac{1}{2}}. \end{aligned}$$

$u(t, x)$ is also bounded in terms of $\|f\|_2$ as follows:

$$|u(t, x)| = \left| \int_{-1}^1 p_t^V(x, y)f(y)dy \right| \leq D_t \int_{-1}^1 |f(y)| dy \leq 2D_t \left(\int_{-1}^1 |f(y)|^2 dy \right)^{\frac{1}{2}}.$$

As one can see by a Cantor diagonalization argument in intervals $I_n = [-1 + \frac{1}{n}, 1 - \frac{1}{n}]$, $T_t = e^{tL^V}$ is a compact self-adjoint operator. Therefore, the spectrum of L^V is discrete.

Note that the lowest eigenvalue has the following variational representation:

$$\lambda = \inf_{\phi} \frac{\int_{-1}^1 (\frac{1}{4}(1-x^4)(\phi'(x))^2 + \frac{1}{2}x^2\phi^2(x))dx}{\int_{-1}^1 \phi^2(x)dx},$$

where the infimum is taken over L^2 equipped with the boundary conditions (9.11). This shows that the spectrum is strictly positive. Let $\{\phi_k(x), \lambda_k\}_{k \in \mathbb{N}}$ be the pair of eigenfunction and eigenvalues such that each $\phi_k(x)$ is odd in x (the even ones do not matter since the initial condition $H_0(x)$ of (9.10) is odd from (9.5)). The solution of (9.10) can be represented as

$$H(x, t) = \sum_{k \in \mathbb{N}} h_k e^{-\lambda_k t} \phi_k(x), \quad (9.13)$$

where

$$h_k = \int_{-1}^1 H_0(x)\phi_k(x)dx.$$

In turn, (9.13) implies that

$$b_n(t) = \sum_{k \in \mathbb{N}} h_k e^{-\lambda_k t} p_n^k \quad (9.14)$$

where p_n^k is defined by

$$\phi_k(x) = \sum_{n=1}^{\infty} (-1)^{n+1} p_n^k x^{2n-1}. \quad (9.15)$$

The p_n^k satisfy the following recurrence relation inherited from (1.11):

$$-\lambda_k p_n^k = (n-1)(n-\frac{1}{2})p_{n-1}^k - n(n+\frac{1}{2})p_{n+1}^k, \quad n \in \mathbb{N}, \quad p_0^k = 0. \quad (9.16)$$

The following lemma describes the asymptotic behavior of np_n .

Lemma 9.6 *For every $\lambda_k > 0$, the recurrence relation in (9.16) implies that*

$$\lim_{n \rightarrow \infty} (2n+1)p_{2n+1}^k = c_k^1 \text{ and } \lim_{n \rightarrow \infty} (2n)p_{2n}^k = c_k^2 \quad (9.17)$$

where c_k^1 and c_k^2 are nonzero constant whose sign is the same as that of p_1^k .

Proof. Assume $p_1 > 0$ and write (9.16) as

$$p_{n+1}^k = \frac{\lambda_k}{n(n+\frac{1}{2})} p_n^k + \frac{(n-1)(n-\frac{1}{2})}{n(n+\frac{1}{2})} p_{n-1}^k \quad n \in \mathbb{N}, \quad p_0^k = 0.$$

We omit the index k in this proof as it plays no role. For sufficiently large n and C depending only on λ ,

$$p_{n+1} \leq \left(1 - \frac{2}{n} + \frac{C}{n^2}\right) \max\{p_n, p_{n-1}\}.$$

This implies $\{p_n\}$ is bounded. More is true:

$$p_m \leq \left[\prod_{l=n}^m \left(1 - \frac{2}{l} + \frac{C}{l^2}\right) \right]^{\frac{1}{2}} \max\{p_n, p_{n-1}\}.$$

This implies that

$$mp_m \leq \exp \left\{ \log m + \frac{1}{2} \sum_{l=n}^m \log \left(1 - \frac{2}{l} + \frac{C}{l^2}\right) \right\} \max\{p_n, p_{n-1}\},$$

which implies further that $\limsup mp_m < \infty$. On the other hand,

$$p_{n+1} \geq \left(1 - \frac{2}{n+1}\right) p_{n-1},$$

which implies

$$p_m \geq \left[\prod_{l=n}^m \left(1 - \frac{2}{l+1}\right) \right]^{\frac{1}{2}} p_n.$$

This implies that

$$mp_m \geq \exp \left\{ \log m - \frac{1}{2} \sum_{l=n}^m \log \left(1 - \frac{2}{l+1} \right) \right\} p_n \quad \text{if } n-m=0 \bmod 2.$$

Thus, $\liminf mp_m > 0$. To show that the sequences in the theorem are Cauchy simply compute

$$|(n+1)p_{n+1} - (n-1)p_{n-1}| = \left| \frac{\lambda}{n(n+1)} (n+1)p_{n+1} + (n-1)p_{n-1} \left[-\frac{1}{2} \frac{1}{n(n+\frac{1}{2})} \right] \right|.$$

Using the fact that $\{np_n\}$ is bounded in n and summing over n we see that the sequence is Cauchy and have proved the lemma. \square

Going back to the proof of Theorem 9.4, using (9.17) in (9.14) implies (9.7) with

$$\bar{C}_1(t) = \sum_{k \in \mathbb{N}} e^{-\lambda_k t} h_k c_k^1 \text{ and } \bar{C}_2(t) = \sum_{k \in \mathbb{N}} e^{-\lambda_k t} h_k c_k^2.$$

Note that there is a $T > 0$ so that $\bar{C}_1(t)\bar{C}_2(t) > 0$ for all $t \geq T$. Finally, (9.7) implies that

$$\frac{1}{2} \frac{d}{dt} \sum_{n=1}^{\infty} b_n^2(t) = - \lim_{N \rightarrow \infty} N(N + \frac{1}{2}) b_N(t) b_{N+1}(t) = -\bar{C}_1(t)\bar{C}_2(t) < 0$$

for all $t \geq T$ which proves (9.8). \square

We also consider the system of forced equations

$$\dot{b}_n = (n-1)(n-\frac{1}{2})b_{n-1} - n(n+\frac{1}{2})b_{n+1} + f(t)\mathbf{1}_{n=m}, \quad (9.18)$$

for $n = 1, 2, \dots$ with boundary condition $b_0(t) = 0$ for all t and $f(t)$ is either a constant forcing term, $f(t) = 1$, or a white-noise process, $f(t) = \dot{W}(t)$.

We have

Theorem 9.7 Consider (9.18) with $f(t) = 1$, and initial condition $b_n(0)$ satisfying (9.1). Then

$$\lim_{t \rightarrow \infty} b_n(t) = b_n^* \equiv \sum_{k \in \mathbb{N}} \frac{d_k p_n^k}{\lambda_k}$$

where p_n^k is defined by (9.15) and

$$d_k = \int_{-1}^1 \phi_k(x) x^m dx$$

In particular, b_n^* satisfies

$$\lim_{n \rightarrow \infty} (2n+1)b_{2n+1}^* = C_1^* > 0 \text{ and } \lim_{n \rightarrow \infty} (2n)b_{2n}^* = C_2^* > 0.$$

Theorem 9.8 Consider (9.18) with $f(t) = \dot{W}(t)$, and initial condition $b_n(-T)$ satisfying (9.1). Then

$$\lim_{T \rightarrow \infty} b_n(t) = b_n^{**}(t) \equiv \sum_{k \in \mathbb{N}} d_k p_n^k \int_{-\infty}^t e^{-\lambda_k s} dW(s)$$

where p_n^k is defined by (9.15).

In particular, $b^{**}(t)$ is a Gaussian process with mean zero and covariance

$$\mathbb{E}b_n^{**}(t)b_m^{**}(t) = \sum_{k,k' \in \mathbb{N}} \frac{d_k d_{k'} p_n^k p_m^{k'}}{\lambda_k + \lambda'_k}$$

and we have

$$\lim_{n \rightarrow \infty} (2n+1)^2 \mathbb{E}(b_{2n+1}^{**}(t))^2 = C_1^{**} > 0 \text{ and } \lim_{n \rightarrow \infty} (2n)^2 \mathbb{E}(b_{2n}^{**}(t))^2 = C_2^{**} > 0.$$

Appendix A Estimates on Taylor coefficients

For the reader's convince, we now state a theorem on the asymptotic of Taylor's series which can be found in [FO90]. .

Theorem A.1 *Let $\Delta(\zeta, \eta, \theta)$ be as in (5.2). Assume that $f(z)$ is analytic in $\Delta(\zeta, \eta, \theta) \setminus \{\zeta\}$ for some $\zeta \in \mathbb{C}$, $\eta > 0$, and $0 < \theta < \pi/2$. If*

$$f(z) \sim \frac{K}{(\zeta - z)^\alpha} \quad \text{as} \quad z \rightarrow \zeta$$

for some $K > 0$ and $\alpha \notin \{0, -1, -2, \dots\}$ then

$$f_n \sim \frac{K}{\Gamma(\alpha)} \frac{n^{\alpha-1}}{\zeta^{n+\alpha}}$$

where f_n is the n -th Taylor coefficient of $f(z)$ about $z = 0$.

Acknowledgements

We thank Percy Deift, Charles Fefferman, Stephanos Venakides and Xin Zhou for useful conversations. J. M. is supported in part by the Sloan Foundation and by an NSF CAREER award. T. S is supported in part by NSF grant DMS05-53403. E. V.-E. is supported in part by NSF grants DMS02-09959 and DMS02-39625, and by ONR grant N00014-04-1-0565.

Appendix References

- [CET94] Peter Constantin, Weinan E, and Edriss S. Titi. Onsager's conjecture on the energy conservation for solutions of Euler's equation. *Comm. Math. Phys.*, 165(1):207–209, 1994.
- [DR00] Jean Duchon and Raoul Robert. Inertial energy dissipation for weak solutions of incompressible Euler and Navier-Stokes equations. *Nonlinearity*, 13(1):249–255, 2000.
- [E01] Weinan E. Stochastic hydrodynamics. In *Current developments in mathematics, 2000*, pages 109–147. Int. Press, Somerville, MA, 2001.
- [Eyi01] Gregory L. Eyink. Dissipation in turbulent solutions of 2D Euler equations. *Nonlinearity*, 14(4):787–802, 2001.
- [Fel54] William Feller. The general diffusion operator and positivity preserving semi-groups in one dimension. *Ann. of Math. (2)*, 60:417–436, 1954.
- [FGV01] G. Falkovich, K. Gawędzki, and M. Vergassola. Particles and fields in fluid turbulence. *Rev. Modern Phys.*, 73(4):913–975, 2001.
- [FO90] Philippe Flajolet and Andrew Odlyzko. Singularity analysis of generating functions. *SIAM J. Discrete Math.*, 3(2):216–240, 1990.

- [Fri95] Uriel Frisch. *Turbulence*. Cambridge University Press, Cambridge, 1995. The legacy of A. N. Kolmogorov.
- [Hil01] Titus Hilberdink. A Tauberian theorem for power series. *Arch. Math. (Basel)*, 77(4):354–359, 2001.
- [McK56] Henry P. McKean, Jr. Elementary solutions for certain parabolic partial differential equations. *Trans. Amer. Math. Soc.*, 82:519–548, 1956.
- [Sri05] Ravi Srinivasan. Simple models with cascade of energy and anomalous dissipation. In Oliver Buhler and Charles Doering, editors, *Fast times and fine scales*, Woods Hole Oceanographic Institution Technical Reports. Woods Hole Oceanographic Institution, 2005.